Name ______________________ Student Number ______________

All solutions are to be presented on the paper in the space provided. The exam is closed book, no calculators. Time for the exam is 75 minutes.

(1) State the derivatives of the following functions:**1 mark each - no part marks**

(a) x^n
nx^{n-1}

(b) $\dfrac{1}{x^n}$
$-\dfrac{n}{x^{n+1}}$ **This question will not count because of an error in the original solutions**

(c) $\sin(x)$
$\cos(x)$

(d) $\cos^{-1}(x)$
$-\dfrac{1}{\sqrt{1-x^2}}$

(e) a^x
$a^x \ln a$

(f) $\log_a x$
$\dfrac{1}{x \ln a}$

(g) $\csc(x)$
$-\csc(x)\cot(x)$

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(h) $\sqrt[a]{x^n}$
$\frac{n}{a}x^{\frac{n}{a}-1}$

(i) $\cot(1)$
0, since $\cot(1)$ is just a number.

(j) e^{cx}
ce^{cx}

(2) Compute the derivatives of the following functions: **2 marks each - 1 mark for correct method, 1 mark for correct answer**

(a) $f(x) = (4 - x^{\frac{2}{5}})^{-\frac{5}{2}}$

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(4 - x^{\frac{2}{5}}\right)^{-\frac{5}{2}} \\
&= -\frac{5}{2}\left(4 - x^{\frac{2}{5}}\right)^{-\frac{7}{2}} \frac{d}{dx}\left(4 - x^{\frac{2}{5}}\right) \\
&= -\frac{5}{2}\left(4 - x^{\frac{2}{5}}\right)^{-\frac{7}{2}} \left(-\frac{2}{5}x^{-\frac{3}{5}}\right) \\
&= \left(4 - x^{\frac{2}{5}}\right)^{-\frac{7}{2}} x^{-\frac{3}{5}}
\end{aligned}$$

(b) $f(x) = \sqrt{\frac{1-x^2}{1+x^2}}$

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \\
&= \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}} \left(\frac{-2x(1+x^2)-(1-x^2)2x}{(1+x^2)^2}\right)
\end{aligned}$$

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$$\begin{aligned}
&= \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left(\frac{-4x}{(1+x^2)^2}\right) \\
&= \frac{1}{2}\left(\frac{1+x^2}{1-x^2}\right)^{\frac{1}{2}}\left(\frac{-4x}{(1+x^2)^2}\right) \\
&= \frac{-2x}{(1-x^2)^{\frac{1}{2}}(1+x^2)^{\frac{3}{2}}}
\end{aligned}$$

(c) $f(x) = \log_3(x^2+1)$

$$f'(x) = \frac{2x}{(x^2+1)\ln 3}$$

(d) $f(x) = \sin^{-1}(-x^2)$

$$\begin{aligned}
f'(x) &= \left(\frac{1}{\sqrt{1-(-x^2)^2}}\right)(-2x) \\
&= \frac{-2x}{\sqrt{1-x^4}}
\end{aligned}$$

(e) $f(x) = x^{2x}$. Must use logarithmic differentiation. Let $y = x^{2x}$.Then

$$\begin{aligned}
\ln y &= \ln x^{2x} \\
\ln y &= 2x\ln x \\
\frac{d}{dx}\ln y &= \frac{d}{dx}(2x\ln x) \\
\frac{y'}{y} &= 2\ln x + 2x\frac{1}{x} \\
\frac{y'}{y} &= 2\ln x + 2 \\
y' &= y(2\ln x + 2) \\
y' &= x^{2x}(2\ln x + 2)
\end{aligned}$$

(3) Find the equation of the tangent line to $x^2 + \cos(x+y) + y^2 = \dfrac{\pi^2}{4}$ at $\left(0, \dfrac{\pi}{2}\right)$. **5 marks**

Need to implicitely differentiate.

$$\frac{d}{dx}\left(x^2 + \cos(x+y) + y^2\right) = \frac{d}{dx}\left(\frac{\pi^2}{4}\right)$$
$$2x - \sin(x+y)(1+y') + 2yy' = 0$$

At $\left(0, \dfrac{\pi}{2}\right)$ this becomes

$$0 - \sin\left(\frac{\pi}{2}\right)(1+y') + \pi y' = 0$$
$$-(1+y') + \pi y' = 0$$
$$y' = \frac{1}{\pi - 1}$$

So, the equation of the tangent line is

$$y = \frac{1}{\pi - 1}x + \frac{\pi}{2}$$

(4) Find the derivative of $f(x) = \dfrac{\sqrt{1+x^2}\,(1-x)^2}{\sqrt[3]{2+x^2}\,(3-x^{10})^3}$. Do not simplify. **5 marks. Maximum of 3 marks for not using logarithmic differentiation**

Use logarithmic differentiation to simplify the expression. Let

$$y = \frac{\sqrt{1+x^2}\,(1-x)^2}{\sqrt[3]{2+x^2}\,(3-x^{10})^3}$$

Then

$$\begin{aligned} \ln y &= \ln\left(\frac{\sqrt{1+x^2}\,(1-x)^2}{\sqrt[3]{2+x^2}\,(3-x^{10})^3}\right) \\ &= \frac{1}{2}\ln(1+x^2) + 2\ln(1-x) - \frac{1}{3}\ln(2+x^2) - 3\ln(3-x^{10}) \end{aligned}$$

And implicitely differentiating

$$\begin{aligned} \frac{y'}{y} &= \frac{x}{1+x^2} - \frac{2}{1-x} - \frac{2x}{3(2+x^2)} + \frac{30x^9}{3-x^{10}} \\ y' &= y\left(\frac{x}{1+x^2} - \frac{2}{1-x} - \frac{2x}{3(2+x^2)} + \frac{30x^9}{3-x^{10}}\right) \end{aligned}$$

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(5) State Extreme Value Theorem. Verify the Extreme Value Theorem for $f(x) = x^2$ on $[-1, 3]$. That is, show that $f(x)$ has an absolute maximum and an absolute minimum. **5 marks. 2 marks for stating the theorem *exactly* correctly. 3 marks for verification.**

The extreme value theorem: Let f be a continuous function on $[a, b]$. Then f has an absolute maximum and an absolute minimum in $[a, b]$.

For this particular function, there is a critical point at $x = 0$. Evaluating the function at $x = 0$ and at the endpoints gives $f(0) = 0$, $f(-1) = 1$ and $f(3) = 9$. Therefore the absolute maximum is 9 and the absolute minimum is 0.

(6) Compute the following limits:

(a) $\lim_{x\to 1} \dfrac{\sin(x-1)}{x-1} = 1$. **2 marks, no part marks**.

(b) $\lim_{\theta\to 0} \dfrac{\sin(3\theta)}{\sin(4\theta)}$. **3 marks. No marks if the 3 and the 4 are factored out of the trig. functions, because that would be *completely wrong*.**

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\sin(3\theta)}{\sin(4\theta)} &= \lim_{\theta\to 0} \frac{\frac{\sin(3\theta)}{3\theta}3\theta}{\frac{\sin(4\theta)}{4\theta}4\theta} \\
&= \frac{3}{4} \lim_{\theta\to 0} \frac{\frac{\sin(3\theta)}{3\theta}}{\frac{\sin(4\theta)}{4\theta}} \\
&= \frac{3}{4}
\end{aligned}$$

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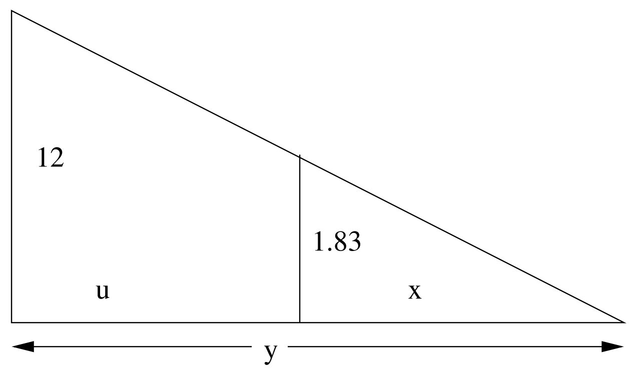
(7) A mob of angry math 110 students stand under a street light mounted at the top of a 12m tall pole. A 1.83m tall math instructor runs as fast as he can away from the street light at a rate of 5m/s. How fast is the tip of his shadow moving when he is 25m from the pole? **5 marks. This is question 9 from the text with some numbers changed, so the marker will be *brutal*.** Let u be the distance of the instructor from



the pole and x be the length of the shadow. Then $\frac{du}{dt} = 5$. We want $\frac{dy}{dt}$ (not $\frac{dx}{dt}$, since that is the rate of change of the length of the shadow). Use similar triangles to write

$$\frac{12}{y} = \frac{1.83}{x}$$

So that

$$12\frac{dx}{dt} = (1.83)\frac{dy}{dt} \tag{1}$$

Since $u = x + y$ we have

$$\frac{du}{dt} + \frac{dx}{dt} = \frac{dy}{dt}$$
$$5 + \frac{dx}{dt} = \frac{dy}{dt}$$
$$\frac{dx}{dt} = \frac{dy}{dt} - 5$$

Plug this into equation(1) to get

$$12(\frac{dy}{dt} - 5) = (1.83)\frac{dy}{dt}$$
$$\frac{dy}{dt} = \frac{60}{12 - 1.83} = \frac{60}{10.17}$$

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